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Balanced Sets and Circuits in a Transversal Space

JEREMY DAWSON

Department of Pure Mathematics, The University of Sheffield, England

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The study of fully dependent sets (unions of circuits) has played a part in characterizing transversal spaces. In fact, the fully dependent sets satisfy $|\Delta(F)| = \rho(F)$ in any deltoid representation, and it is with a consideration of this property that we begin the present paper. We study "balanced" sets and from our results draw conclusions about fully dependent sets and circuits in a transversal space. These include upper bounds for the number of circuits, and the result that a non-trivial transversal space can be neither a hereditary circuit space nor the dual of a geometric hereditary circuit space. The paper is reasonably self-contained; all unusual terms are defined as they are encountered.

1. Balanced Sets

Throughout the paper, all independence spaces and transversal spaces considered are of finite rank. The reader who requires a general background to transversal theory is referred to [6]. In this section, let $\mathscr{E}$ be a transversal structure on E presented by the family $\mathfrak{A} = (A_i : i \in I)$, let (I, Δ, E) be the associated deltoid, and let ρ be the rank function of $(E, \mathscr{E})$. Then, for every $F \subseteq E$, $\rho(F) \leqslant |\Delta(F)|$. A set $F \subseteq E$ is *balanced* (with respect to Δ) if $\rho(F) = |\Delta(F)|$. Easy examples illustrate the dependence on Δ. By Hall's theorem a set $F \subseteq E$ is in $\mathscr{E}$ if and only if $|\Delta(G)| \geqslant |G|$ for all $G \subseteq F$; thus F is a circuit of $\mathscr{E}$ if and only if $|\Delta(G)| \geqslant |G|$ for all $G \subset F$ and $|\Delta(F)| = |F| - 1$ $(= \rho(F))$. So the circuits of $(E, \mathscr{E})$ are examples of balanced sets. We now prove some elementary properties of balanced sets. For $A \subseteq E$, $[A]$ will denote the flat spanned by A in $(E, \mathscr{E})$.

1.1. THEOREM. *Let F, $G \subseteq E$, let F be balanced, and let $[F] = [G]$. Then G is balanced* (*with respect to the same deltoid*).

Proof. As $[F] = [G]$ we have $\rho(F) = \rho(G)$. Also

$$\Delta(F) = \Delta([F]) = \Delta([G]) = \Delta(G)$$

and the result follows. ∎

1.2. COROLLARY. *Let F be balanced, let H be a basis of F, and let $H \subseteq G \subseteq [F]$. Then H and G are balanced.* ▮

1.3. COROLLARY. *Let F be balanced. Then $[F]$ is balanced.* ▮

1.4. THEOREM. *Any union of balanced sets is balanced.*

Proof. Let J be an arbitrary set and let $F = \bigcup_{j \in J} F_j$, where each F_j is balanced. Let G_j be a basis of F_j for each j, and write $G = \bigcup_{j \in J} G_j$. Then, since

$$\rho(G) = \min_{H \subseteq G} \{| \Delta(H)| + | G \backslash H |\},$$

there exists a maximal $H \subseteq G$ with $\rho(G) = | \Delta(H)| + | G \backslash H |$. For each $j \in J$, $H \cap G_j \in \mathscr{E}$ and, by 1.2, G_j is balanced. Thus

$$\begin{aligned}
\rho(G) &\leqslant | \Delta(H \cup G_j)| + | G \backslash (H \cup G_j)| \\
&= | \Delta(H)| + | \Delta(G_j)| - | \Delta(H) \cap \Delta(G_j)| + | G \backslash H | - | G_j | + | H \cap G_j | \\
&\leqslant | \Delta(H)| + | G_j | - | \Delta(H \cap G_j)| + | G \backslash H | - | G_j | + | H \cap G_j | \\
&\leqslant | \Delta(H)| + | G \backslash H | = \rho(G).
\end{aligned}$$

Hence, by the maximality of H,

$$G = \bigcup_{j \in J} G_j \subseteq H \subseteq G;$$

which shows that $H = G$ and that

$$\rho(G) = | \Delta(G)| + | G \backslash G | = | \Delta(G)|.$$

Thus G is balanced and, as $G \subseteq F \subseteq [G]$, it follows from 1.2 that F is balanced.

1.5. THEOREM. *Let F be balanced. Then $[F] = E \backslash \Delta(I \backslash \Delta(F))$.*

Proof. Let G be a basis of F. Then

$$[F] = F \cup \{e \in E \colon \{e\} \cup G \notin \mathscr{E}\}.$$

Since F is balanced we know that $\rho(F) = | G | = | \Delta(F)|$, and so G is a transversal of $(A_i : i \in \Delta(F))$. So if $e \in \Delta(I \backslash \Delta(F))$ it follows that $\{e\} \cup G \in \mathscr{E}$ and so $e \notin [F]$. Conversely, if $e \notin \Delta(I \backslash \Delta(F))$ and $e \notin G$, then

$$| \Delta(\{e\} \cup G)| = | \Delta(G)| = | G | < |\{e\} \cup G |;$$

and so $\{e\} \cup G \notin \mathscr{E}$ and $e \in [F]$. ▮

1.6. COROLLARY. *Let $F, G \subseteq E$ and let F be balanced with $\Delta(G) \subseteq \Delta(F)$. Then $G \subseteq [F]$.*

Proof. Since $|\varDelta(F)| = \rho(F)$ and $\varDelta(G) \subseteq \varDelta(F)$, it follows that

$$\rho(F \cup G) \leqslant |\varDelta(F \cup G)| = |\varDelta(F)| = \rho(F);$$

and $G \subseteq [F]$ as required. ▌

Not every contraction of a transversal structure is itself transversal; but we have the following result.

1.7. THEOREM. *Let F be balanced. Then the contraction of $\mathscr{E}$ to $E\backslash F$ is a transversal structure on $E\backslash F$.*

Proof. Recall (from [7], for example) that the contraction of $\mathscr{E}$ to $E\backslash F$ (or "away from F") is the collection

$$\{X \subseteq E\backslash F : X \cup H \in \mathscr{E} \text{ for some basis } H \text{ of } F\};$$

and this is at once seen to be the transversal structure of $(A_i : i \in I\backslash\varDelta(F))$. ▌

1.8. THEOREM. *Let $F_1, \ldots, F_k$ be balanced with $F_i \cup F_j \in \mathscr{E}$ for each i, j. Then $\bigcap_{1\leqslant i\leqslant k} F_i$ is balanced and $\bigcup_{1\leqslant i\leqslant k} F_i \in \mathscr{E}$.*

Proof. For $k = 2$, since $F_1 \cup F_2 \in \mathscr{E}$ and F_1, F_2 are balanced,

$$\begin{aligned} |F_1| + |F_2| &= |\varDelta(F_1)| + |\varDelta(F_2)| \\ &= |\varDelta(F_1) \cup \varDelta(F_2)| + |\varDelta(F_1) \cap \varDelta(F_2)| \\ &\geqslant |\varDelta(F_1 \cup F_2)| + |\varDelta(F_1 \cap F_2)| \\ &\geqslant |F_1 \cup F_2| + |F_1 \cap F_2| \\ &= |F_1| + |F_2| \end{aligned}$$

and so there is equality throughout. Hence

$$|\varDelta(F_1 \cap F_2)| = |\varDelta(F_1) \cap \varDelta(F_2)| = |F_1 \cap F_2| = \rho(F_1 \cap F_2)$$

and $F_1 \cap F_2$ is balanced. An easy inductive argument shows that

$$|\varDelta(F_1 \cap \cdots \cap F_k)| = |\varDelta(F_1) \cap \cdots \cap \varDelta(F_k)| = |F_1 \cap \cdots \cap F_k| = \rho(F_1 \cap \cdots \cap F_k)$$

and so $\bigcap_{1\leqslant i\leqslant k} F_i$ is balanced. From 1.4, $F_1 \cup \cdots \cup F_k$ is balanced, and a standard inclusion–exclusion argument now shows that

$$\rho(F_1 \cup \cdots \cup F_k) = |\varDelta(F_1 \cup \cdots \cup F_k)| = |\varDelta(F_1) \cup \cdots \cup \varDelta(F_k)| = |F_1 \cup \cdots \cup F_k|$$

as required. ▌

In fact, 1.8 applies equally to any family of balanced sets, because it is easy to check that any family of sets satisfying the hypothesis of 1.8 can possess at most $2^{\rho(E)}$ distinct sets. Also, 1.8 extends to the case when $F_1,\dots,F_k$ are unions of balanced sets, $F_i = \bigcup_{j\in J_i} G_{ij}$ say, with $G_{ij} \cup G_{i'j'} \in \mathscr{E}$ for each $i \neq i'$. We omit the proof of this generalization.

1.9. THEOREM. *Let $F_1,\dots,F_k$ be balanced. Then*

$$\sum_{J\subseteq\{1,\dots,k\}} (-1)^{|J|+1}\rho\Big(\bigcup_{j\in J} F_j\Big) \geqslant \rho\Big(\bigcap_{1\leqslant j\leqslant k} F_j\Big).$$

Proof.

$$\begin{aligned}\sum_{J\subseteq\{1,\dots,k\}} (-1)^{|J|+1}\rho\Big(\bigcup_{j\in J} F_j\Big) &= \sum_{J\subseteq\{1,\dots,k\}} (-1)^{|J|+1}\Big|\Delta\Big(\bigcup_{j\in J} F_j\Big)\Big| \\ &= \sum_{J\subseteq\{1,\dots,k\}} (-1)^{|J|+1}\Big|\bigcup_{j\in J}\Delta(F_j)\Big| \\ &= \Big|\bigcap_{1\leqslant j\leqslant k}\Delta(F_j)\Big| \\ &\geqslant \Big|\Delta\Big(\bigcap_{1\leqslant j\leqslant k} F_j\Big)\Big| \geqslant \rho\Big(\bigcap_{1\leqslant j\leqslant k} F_j\Big). \quad \blacksquare\end{aligned}$$

Note that 1.9 is equivalent to the apparently weaker result for balanced flats because, if $F_1,\dots,F_k$ are balanced, then so are $[F_1],\dots,[F_k]$, and any unions of them; thus

$$\sum_{J\subseteq\{1,\dots,k\}} (-1)^{|J|+1}\rho\Big(\bigcup_{j\in J} F_j\Big) = \sum_{J\subseteq\{1,\dots,k\}} (-1)^{|J|+1}\rho\Big(\bigcup_{j\in J} [F_j]\Big)$$

and

$$\rho\Big(\bigcap_{1\leqslant j\leqslant k} [F_j]\Big) \geqslant \rho\Big(\bigcap_{1\leqslant j\leqslant k} F_j\Big).$$

The inequality of 1.9 applied to a subcollection of the balanced sets in fact characterizes transversal spaces, as we shall remark in the next section.

2. FULLY DEPENDENT SETS

In this section $(E, \mathscr{E})$ is an independence space (of finite rank) upon which we impose no particular restraints except when stated. Recall from [5] that $F \subseteq E$ is *fully dependent* if F is a union of circuits (or, equivalently if $f \in [F\backslash\{f\}]$ for each $f \in F$). In particular, the empty set is fully dependent (but every other fully dependent set is a dependent set). We remarked at the beginning of Section 1 that circuits in a transversal space are balanced sets and so it is an immediate consequence of 1.4 that fully dependent sets in transversal space are balanced. So, of course, all the results of Section 1 apply in particular to the fully dependent sets in a transversal space. In fact the inequality of 1.9 applied to the fully dependent sets characterizes such spaces [5]. Another well-known characterization of transversal spaces is in terms of their hyperplanes [4], and it is very easy to prove in a similar way an analogous characterization in terms of fully dependent sets. For the sake of completeness we give a short proof of this result derived from the hyperplane result of [4].

2.1. THEOREM. *Let $(E, \mathscr{E})$ be an independence space of rank n. Then $(E, \mathscr{E})$ is a transversal space if and only if there exists a family $(F_1, \ldots, F_n)$ of fully dependent sets with $\rho(F_{i_1} \cap \cdots \cap F_{i_j}) \leqslant n - j$ for each $\{i_1, \ldots, i_j\}_{\neq} \subseteq \{1, \ldots, n\}$ and such that, given a circuit C, it is contained in F_i for at least $n - \rho(C)$ values of i, $1 \leqslant i \leqslant n$.*

Proof. Assume first that $(E, \mathscr{E})$ is a transversal space and that (by [4]) $H_1, \ldots, H_n$ are hyperplanes with $\rho(H_{i_1} \cap \cdots \cap H_{i_j}) \leqslant n - j$ for each $\{i_1, \ldots, i_j\}_{\neq} \subseteq \{1, \ldots, n\}$ and such that, given a circuit C, it is contained in at least $n - \rho(C)$ of the H_i's. For each i, $1 \leqslant i \leqslant n$, let F_i be the union of all the circuits contained in H_i. Then $F_1, \ldots, F_n$ are fully dependent sets (and, in fact, are flats), and it is easy to check that they have the required properties.

Conversely, assume that such $F_1, \ldots, F_n$ exist and let τ be the transversal structure of the family $(E \backslash F_1, \ldots, E \backslash F_n)$. Then for $A \in \mathscr{E}$ with $|A| = n$ and for $\{i_1, \ldots, i_j\}_{\neq} \subseteq \{1, \ldots, n\}$ we have

$$\begin{aligned} |A \cap (E \backslash F_{i_1} \cup \cdots \cup E \backslash F_{i_j})| &= n - |A \cap F_{i_1} \cap \cdots \cap F_{i_j}| \\ &\geqslant n - \rho(F_{i_1} \cap \cdots \cap F_{i_j}) \geqslant j \end{aligned}$$

and so, by Halls theorem, $A \in \tau$. Thus $\mathscr{E} \subseteq \tau$. Assume that $\mathscr{E} \not\subseteq \tau$ and choose a $B \in \tau \backslash \mathscr{E}$ of least cardinality. Then B is a circuit of $(E, \mathscr{E})$, of rank r say, and is contained in $F_1, \ldots, F_{n-r}$ say. Hence $B \cap (E \backslash F_i) \neq \varnothing$ for at most $r (< |B|)$ i's; which contradicts the fact that $B \in \tau$. Thus $\mathscr{E} = \tau$, and $\mathscr{E}$ is a transversal structure as required. ∎

2.2. COROLLARY. *Let $(E, \mathscr{E})$ be an independence space of rank $n > 2$ all of whose circuits are of rank n, 1, or 0, and such that, if $C_1, \ldots, C_r$ are all the distinct circuits of rank 1, then the family $([C_1], \ldots, [C_r])$ contains at most n distinct members. Then $(E, \mathscr{E})$ is a transversal space.*

Proof. If $(E, \mathscr{E})$ has no circuits of rank 1, then, by 2.1 it is clearly a transversal space. So assume that $C_1, \ldots, C_s$ $(s \leqslant n)$ is a collection of distinct circuits of rank 1 which contain bases of all the flats of rank 1 spanned by circuits, and no two of which span the same flat. Then it is easy to check that, since there are no circuits of rank $2, \ldots, n-1$,

$$[C_1 \cup \cdots \cup C_{s-1}] = [C_1] \cup \cdots \cup [C_{s-1}]$$

and so on, and that the sets

$$F_1 = [C_2 \cup \cdots \cup C_s], \quad F_2 = [C_1 \cup C_3 \cup \cdots \cup C_s], \ldots,$$
$$F_s = [C_1 \cup \cdots \cup C_{s-1}], \quad F_{s+1} = \cdots = F_n = [C_1 \cup \cdots \cup C_s]$$

are fully dependent and satisfy the conditions of 2.1. ∎

2.3. COROLLARY. *Let $(E, \mathscr{E})$ be an independence space of rank n with at most two circuits of rank less than n. Then $(E, \mathscr{E})$ is a transversal space.*

Proof. If $(E, \mathscr{E})$ has no circuits of rank less than n, then, by 2.1, it is clearly a transversal space. If C, of rank r say, is the only circuit of rank less than n, then the sets

$$F_1 = \cdots = F_r = \varnothing, \qquad F_{r+1} = \cdots = F_n = C$$

satisfy the conditions of 2.1. Finally, if C_1, C_2 are the circuits of rank r, s ($<n$), respectively, then either C_1 and C_2 are disjoint or $C_1 \cup C_2$ contains a circuit of rank n and

$$\begin{aligned}
F_i &= C_1 && (i \in \{1,\ldots, n-r\} \cap \{1,\ldots, s\}),\\
F_i &= C_2 && (i \in \{n-r+1,\ldots, n\} \cap \{s+1,\ldots, n\}),\\
F_i &= C_1 \cup C_2 && (i \in \{1,\ldots, n-r\} \cap \{s+1,\ldots, n\}),\\
F_i &= \varnothing && (i \in \{n-r+1,\ldots, n\} \cap \{1,\ldots, s\})
\end{aligned}$$

defines a collection of fully dependent sets satisfying 2.1. ∎

The remark concerning flats made in the proof of 2.1 shows that the same result holds with fully dependent sets replaced by fully dependent flats. For the last result of this section we assume that (E, $\mathscr{E}$) is a transversal space, and that (I, Δ, E) is the deltoid associated with a finite presentation $(A_i : i \in I)$ of $(E, \mathscr{E})$ where the complements of the A_i's are fully dependent flats.

2.4. THEOREM. *Let $F \subseteq E$ be balanced and contained in a fully dependent flat, and let $F_1, \ldots, F_r$ be the fully dependent flats containing F. Then $[F] = \bigcap_{1 \leqslant i \leqslant r} F_i$.*

Proof. By 1.5, $F \subseteq E \backslash A_j$ for each $j \in I \backslash \Delta(F)$, and by the choice of presentation each $E \backslash A_j$ is a fully dependent flat. Thus, again by 1.5,

$$\bigcap_{1 \leqslant i \leqslant r} F_i \subseteq \bigcap_{j \in I \backslash \Delta(F)} E \backslash A_j = E \backslash \Delta(I \backslash \Delta(F)) = [F] \subseteq \bigcap_{1 \leqslant i \leqslant r} F_i;$$

and the result follows. ∎

3. Circuits and Circuit Spaces

We say that a set G in an independence space $(E, \mathscr{E})$ is a *circuit basis* if $G \cup \{g\}$ is a circuit for some $g \in E \backslash G$. In [2] a *hereditary circuit space* is defined as an independence space in which each independent 2-set is a circuit basis; and it is mentioned there that, in such a space, each independent r-set, $r \geqslant 2$, is a circuit basis. In particular, we shall need the result that a hereditary circuit space has no independent flats of cardinality 2 or more. We recall also that a *circuit space* [1, 7] is an independence space in which each basis is a circuit basis, and we remark that a hereditary circuit space of rank at least 2 is necessarily a circuit space.

Note that in a transversal space circuit bases are balanced; this is immediate from 1.2 and the fact that circuits are balanced. We shall continue to use without further comment $(A_i : i \in I)$ as a presentation of a transversal space and Δ as the associated deltoid.

3.1. Theorem. *Let $(E, \mathscr{E})$ be a transversal space of rank n. Then, for each r satisfying $2 \leqslant r \leqslant n$, there exists an independent r-set which is not a circuit basis.*

Proof. Let $2 \leqslant r \leqslant n$. We may choose the presentation $(A_i : i \in I)$ of $(E, \mathscr{E})$ with $|I| = \rho(E)$, $I = \{1,..., n\}$ say. If $|\Delta(\{g\})| = 1$ for each $\{g\} \in \mathscr{E}$, then the $A_1 ,..., A_n$ are pairwise disjoint and, since $r \geqslant 2$, there are no $(r + 1)$-circuits. So assume, say, that $|\Delta(\{g_1\})| > 1$ for some $\{g_1\} \in \mathscr{E}$ and that $\{g_1 ,..., g_n\}$ is a basis of $\mathscr{E}$ with $g_1 \in A_1 \cap A_n$, $g_2 \in A_2 ,..., g_n \in A_n$. Then

$$\{1,..., r, n\} \subseteq \Delta(\{g_1 ,..., g_r\})$$

and, since $r < n$,

$$|\{g_1 ,..., g_r\}| = r < r + 1 \leqslant |\Delta(\{g_1 ,..., g_r\})|$$

which shows that $\{g_1 ,..., g_r\}$ is not balanced and hence not a circuit basis. ∎

3.2. Corollary. *A transversal space of rank $n > 2$ cannot be a hereditary circuit space.* ∎

3.3. Theorem. *Let F be an $(n + 1)$-circuit in a transversal space. Then, for $1 \leqslant r < n$, the number of balanced r-subsets G of F is equal to the number of $(n - r)$-subsets K of $\Delta(F)$ with $|\Delta(K) \cap F| = |K| + 1$.*

Proof. Let $J = \Delta(F)$ and let (J, Δ', F) be the induced subdeltoid of (I, Δ, E) (i.e., $\Delta' = \Delta \cap (J \times F)$). Then, as F is balanced, $|J| = |F| - 1 = n$. Let $1 \leqslant r < n$ and let $\mathscr{A}$ be the collection of balanced r-subsets G of F (i.e., with $|\Delta'(G)| = |G|$) and let $\mathscr{B}$ be the collection of $(n-r)$-subsets K of J with $|\Delta'(K)|$ $(= |\Delta(K) \cap F|) = |K| + 1$. For each $G \in \mathscr{A}$ let

$$\Phi(G) = J\backslash\Delta'(G).$$

Then, for $G \in \mathscr{A}$, $\Phi(G) \subseteq J$ and $|\Phi(G)| = |J| - |\Delta'(G)| = n - r$. Further,

$$\Delta'(\Phi(G)) = \Delta'(J\backslash\Delta'(G)) \subseteq F\backslash G;$$

also $\Delta'(F\backslash\Delta'(\Phi(G))) \subseteq J\backslash\Phi(G)$ and $F\backslash\Delta'(\Phi(G)) \in \mathscr{E}$: from which we deduce that

$$\begin{aligned} r = n - |\Phi(G)| = |J\backslash\Phi(G)| &\geqslant |\Delta'(F\backslash\Delta'(\Phi(G)))| \\ &\geqslant |F\backslash\Delta'(\Phi(G))| = n + 1 - |\Delta'(\Phi(G))| \end{aligned}$$

and $|\Delta'(\Phi(G))| \geqslant n - r + 1 = |F\backslash G|$. Hence $\Delta'(\Phi(G)) = F\backslash G$ and $|\Delta'(\Phi(G))| = n - r + 1 = |\Phi(G)| + 1$. It follows that $\Phi(G) \in \mathscr{B}$ and $\Phi\colon \mathscr{A} \to \mathscr{B}$.

Similarly, for each $K \in \mathscr{B}$, let

$$\Psi(K) = F\backslash\Delta'(K).$$

Then, for $K \in \mathscr{B}$, $\Psi(K) \in \mathscr{E}$ and, as above,

$$\Delta'(\Psi(K)) = \Delta'(F\backslash\Delta'(K)) \subseteq J\backslash K$$

and

$$|\Delta'(\Psi(K))| \geqslant |\Psi(K)| = r = |J\backslash K|.$$

Hence $|\Delta'(\Psi(K))| = |\Psi(K)| = r$, $\Psi\colon \mathscr{B} \to \mathscr{A}$ and $\Delta'(\Psi(K)) = J\backslash K$.

Finally, for $G \in \mathscr{A}$ and $K \in \mathscr{B}$,

$$\Psi(\Phi(G)) = F\backslash\Delta'(\Phi(G)) = F\backslash(F\backslash G) = G$$

and

$$\Phi(\Psi(K)) = J\backslash\Delta'(\Psi(K)) = J\backslash(J\backslash K) = K.$$

Hence Φ, Ψ are inverse functions and $|\mathscr{A}| = |\mathscr{B}|$; which is the required result. ∎

3.4. Corollary. *Let F be an $(n+1)$-circuit in a transversal space and let $1 \leqslant r < n$. Then the number of r-subsets of F which are circuit bases is at most* $\binom{n}{r}$.

Proof. The number of r-subsets of F which are circuit bases is less than or equal to the number of balanced r-subsets of F. But, by 3.3, this is less than or equal to the number of $(n-r)$-subsets of the n-set $\Delta(F)$. ∎

3.5. COROLLARY (cf. 3.1.). *If a transversal space of rank n is a circuit space and $1 \leqslant r < n$, then not every independent r-set is a circuit basis.*

Proof. Apply 3.4 to any of the $(n+1)$-circuits. ▌

A natural question arising from 3.2 and the work on circuit spaces in [1] is whether the dual of a transversal space can be a hereditary circuit space. For the remainder of this paper $(E, \mathscr{E})$ will be an independence space in which each doubleton is independent (i.e., a geometry), and where E is assumed finite. We shall show that no such space of rank greater than 2 can be both a hereditary circuit space and the dual of a transversal space. In the proof we make use of the striking result from [3] that in a geometry of rank greater than 2 the number of flats of rank 2 is at least equal to the number of points. The dual space of $(E, \mathscr{E})$ will be denoted by $(E, \mathscr{E}^*)$ with rank function ρ^*.

3.6. LEMMA. *Let A be a circuit in $(E, \mathscr{E})$ and let $\{x, y\}_{\neq} \subseteq E$. If $\{x, y\} \nsubseteq [A]$ or $|A| > 3$, then there is a hyperplane H of $(E, \mathscr{E})$ with $A \nsubseteq H$ and $\{x, y\} \subseteq H$.*

Proof. If, for each $a \in A$, either $a \in \{x, y\}$ or $\{x, y, a\}_{\neq}$ is dependent, then $A \subseteq [x, y]$ and $|A| \leqslant 3$. Since each doubleton of E is independent it follows that $|A| = 3$, $\rho(A \cup \{x, y\}) = 2$ and $\{x, y\} \subseteq [A]$. So, if $\{x, y\} \nsubseteq [A]$ or $|A| > 3$, then we can deduce that $\{x, y, a\}_{\neq} \in \mathscr{E}$ for some $a \in A$. Extend $\{x, y, a\}$ to a basis B of $(E, \mathscr{E})$ and let $H = [B \backslash \{a\}]$. Then $\{x, y\} \subseteq B \backslash \{a\} \subseteq H$ and $a \in A \backslash H$. ▌

3.7. LEMMA. *Let $(E, \mathscr{E})$ be a hereditary circuit space and let F be fully dependent in $(E, \mathscr{E}^*)$ with F containing a basis of $(E, \mathscr{E}^*)$. Then $|E \backslash F| \leqslant 1$.*

Proof. Note that F is fully dependent (a union of circuits) in $\mathscr{E}^*$ if and only if $E \backslash F$ is a flat (an intersection of hyperplanes) in $\mathscr{E}$. Also if F contains a basis of $\mathscr{E}^*$, then $E \backslash F$ is contained in a basis of $\mathscr{E}$. Thus $E \backslash F$ is an independent flat in a hereditary circuit space and so, by our earlier comments, $|E \backslash F| \leqslant 1$. ▌

3.8. THEOREM. *Let $(E, \mathscr{E})$ be a hereditary circuit space of rank greater than 2. Then $(E, \mathscr{E}^*)$ is not a transversal space.*

Proof. Assume otherwise. Let $(E, \mathscr{E})$ have rank n, $(E, \mathscr{E}^*)$ have rank m, and let $(A_1, \ldots, A_m)$ be a presentation of $(E, \mathscr{E}^*)$ with each $E \backslash A_i$ a hyperplane of $(E, \mathscr{E}^*)$ (i.e., each A_i is a circuit of $(E, \mathscr{E})$). Let $\{x, y\}_{\neq} \subseteq E$. We shall show that there exists an A_i with $|A_i| = 3$ and $\{x, y\} \subseteq [A_i]$ (where the notation [ ] still applies to $(E, \mathscr{E})$). For, if not, then by 3.6 there exist hyperplanes $H_1, \ldots,$ H_m in $(E, \mathscr{E})$ with $A_i \nsubseteq H_i$ and $\{x, y\} \subseteq H_i$ for each i. But then $F = \bigcup_{1 \leqslant i \leqslant m}$ $E \backslash H_i$ is a union of circuits in $(E, \mathscr{E}^*)$, and so it is balanced in $(E, \mathscr{E}^*)$ (with respect to the associated deltoid of any presentation). Thus, since $F \cap A_i \neq \varnothing$

for each i, we have that $\rho^*(F) = m$ and that F contains a basis of $(E, \mathscr{E}^*)$. So by 3.7, $| E\backslash F | \leqslant 1$; which contradicts the fact that $\{x, y\} \subseteq E\backslash F$. Hence there does indeed exist an A_i with $| A_i | = 3$ and $\{x, y\} \subseteq [A_i]$. Therefore the flats $[A_1],..., [A_m]$ include all the flats of rank 2 of $(E, \mathscr{E})$. Thus, in $(E, \mathscr{E})$,

$$\text{the number of flats of rank } 2 \leqslant m < m + n = \text{number of points;}$$

which is a contradiction. The required result follows. ▮

We remark that 3.2 and 3.8 show that no nontrivial transversal space or the dual of one can be linearly represented by the whole of a vector space.

Note added in proof. Dr. P. Vamos has shown that 3.8 and the finite case of 3.2 follow easily from the results that finite transversal spaces are represented over the reals, but that finite hereditary circuit spaces (by Sylvester's theorem) are not.

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